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ON THE MINIMA OF YANG-MILLS FUNCTIONALS

CHAOHAO GU

Dedicated to the author's teacher Professor Buchin Su

In a previous paper [1] we found some lower bounds of the Yang-Mills functional on the tangential bundle over a 4-dimensional oriented manifold among all possible metrics with the Christoffel connections as the gauge potentials [2]. In this paper the results are generalized to vector bundles over a 4-dimensional oriented manifolds, provided the structure group (gauge group) G is compact and its Lie algebra g is nonsimple. Some lower bounds of the Yang-Mills functional are obtained and several cases for which the lower bounds are actually the absolute minimums are listed. In particular it is seen that for the Einstein manifolds or the conformally flat manifolds with zero scalar curvature the Yang-Mills functional attains its absolute minimum among all possible metrics on the manifold and all possible connections on the tangential bundles.

Let G be a compact Lie group, and suppose that its Lie algebra g be the direct sum of two Lie algebras

(1) $$g = g_1 + g_2.$$

An arbitrary element α of g may be written in the form $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 \in g_1$, $\alpha_2 \in g_2$. We define a linear mapping $*$ on the Lie algebra g to itself by

$$*(\alpha_1, \alpha_2) = (\alpha_1, -\alpha_2).$$

Then the following relations evidently hold:

(2) (a) $$*^2 = I,$$

(3) (b) $$[\alpha, *\beta] = ([\alpha_1, \beta_1], [\alpha_2, -\beta_2]) = *([\alpha, \beta]),$$

(4) (c) $$\langle *\alpha, *\beta\rangle = \langle \alpha_1, \beta_1\rangle + \langle -\alpha_2, -\beta_2\rangle = \langle \alpha, \beta\rangle,$$

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where [,] and $\langle\ ,\ \rangle$ are the commutator and the invariant inner product respectively..

Theorem 1. *If g is a Lie algebra, and there exists a linear mapping $*: g \to g$ such that* (a) *and* (b) *hold, then g is the direct sum of the subalgebras, except for the trivial case $* = \pm I$.*

Proof. Let

$$g_1 = \{\alpha + *\alpha | \alpha \in g\}, \quad g_2 = \{\alpha - *\alpha | \alpha \in g\}. \tag{5}$$

Since $*$ is nontrivial, g_1 and g_2 are both nontrivial subspaces.

From (b) it is easily seen that

$$[\alpha + *\alpha, \beta + *\beta] = ([\alpha, \beta] + [*\alpha, \beta]) + *([\alpha, \beta] + [*\alpha, \beta]),$$
$$[\alpha - *\alpha, \beta - *\beta] = ([\alpha, \beta] - [*\alpha, \beta]) - *([\alpha, \beta] - [*\alpha, \beta]).$$

Hence g_1 and g_2 are both subalgebras. Moreover, from (a) and (b) we have

$$[\alpha + *\alpha, \beta - *\beta] = [\alpha, \beta] + [*\alpha, \beta] - [\alpha, *\beta] - [*\alpha, *\beta] = 0,$$

i.e., the elements of g_1 and those of g_2 are commutative. Finally, if $\gamma \in g_1 \cap g_2$, then $*\gamma = \pm\gamma$ which implies that $\gamma = 0$. Hence $g_1 \cap g_2 = \{0\}$, and the theorem is proved.

The mapping $*$ is called the generalized dual operator, since it contains the usual duality in R^4 as a particular case: Let g be the Lie algebra so (4) formed by 4×4 skew-symmetric matrices $L = (l_{ab})$, $a, b = 1, \cdots, 4$. Then g_1 and g_2 are sets of self-dual and antiself-dual matrices respectively, i.e.,

$$L \in g_1 \quad \text{iff} \quad l_{ab} = \tfrac{1}{2}\varepsilon_{abcd} l_{cd},$$
$$L \in g_2 \quad \text{iff} \quad l_{ab} = -\tfrac{1}{2}\varepsilon_{abcd} l_{cd}.$$

The $*$ is the dual operator

$$*L = (l^*_{ab})$$

with

$$l^*_{ab} = \tfrac{1}{2}\varepsilon_{abcd} l_{cd}. \tag{6}$$

Now let M be a 4-dimensional oriented Riemannian manifold, and $E \to M$ a vector bundle over M with structure group G. Let b be a connection or gauge potential on E. In a small patch b is expressed as g-valued 1-form

$$b = b_\lambda(x)dx^\lambda, \tag{7}$$

and the field strength is the g-valued 2-form:

$$\begin{aligned} F &= db + \tfrac{1}{2}[b, b] = \tfrac{1}{2}F_{\lambda\mu}dx^\lambda \wedge dx^\mu \\ &= \tfrac{1}{2}(\partial_\lambda b_\mu - \partial_\mu b_\lambda + [b_\lambda, b_\mu])dx^\lambda \wedge dx^\mu. \end{aligned} \tag{8}$$

Define

$$I_W = \int_M \langle F_{pq}, F^{pq} \rangle dV = \int_M K_W dV, \tag{9}$$

$$I_E = \tfrac{1}{2} \int_M \langle F_{pq}, *F_{rs} \rangle \varepsilon^{pqrs} dV = \int_M K_E dV, \tag{10}$$

$$I_P = \tfrac{1}{2} \int_M \langle F_{pq}, F_{rs} \rangle \varepsilon^{pqrs} dV = \int_M K_P dV, \tag{11}$$

$$I_Q = \int_M \langle F_{pq}, *F^{pq} \rangle dV = \int_M K_Q dV, \tag{12}$$

where I_W is the Yang-Mills functional, and $*$ the generalized dual operator. I_P is the integral of the 2nd Chern class or 1st Pontrjagen class up to a constant factor. I_E is the generalization of the Euler's characteristic number and is also a topological invariant, if the manifold is compact and free of boundary. This is a consequence of the Chern-Weil's theorem [3], since the bilinear form $f(\alpha, \beta) = \langle \alpha, *\beta \rangle$ on $g \times g$ is symmetric and invariant. In general I_Q is not a topological invariant. However, for the tangential bundle $I_P = I_Q$ if the connection is the Christoffel connection of the Riemannian metric.

Define the inner product of two g-valued 2-form at each point of M

$$F \cdot \Phi = \langle F_{pq}, \Phi^{pq} \rangle. \tag{13}$$

It is easy to verify that

$$\begin{aligned} F \cdot F &= F^{**} \cdot F^{**} = F^{*\cdot} \cdot F^{*\cdot} = F^{\cdot *} \cdot F^{\cdot *} = K_W, \\ F \cdot F^{\cdot *} &= F^{*\cdot} \cdot F^{**} = K_P, \\ F \cdot F^{*\cdot} &= F^{\cdot *} \cdot F^{**} = K_Q, \\ F \cdot F^{**} &= F^{*\cdot} \cdot F^{\cdot *} = K_E. \end{aligned} \tag{14}$$

Here the 1st $*$ is the generalized dual operator, and the 2nd $*$ is the operator (6).

Let

$$\begin{aligned} F^{++} &= \tfrac{1}{4}(F^{\cdot\cdot} + F^{*\cdot} + F^{\cdot *} + F^{**}), \\ F^{-+} &= \tfrac{1}{4}(F^{\cdot\cdot} - F^{*\cdot} + F^{\cdot *} - F^{**}), \\ F^{+-} &= \tfrac{1}{4}(F^{\cdot\cdot} + F^{*\cdot} - F^{\cdot *} - F^{**}), \\ F^{--} &= \tfrac{1}{4}(F^{\cdot\cdot} - F^{*\cdot} - F^{\cdot *} + F^{**}). \end{aligned} \tag{15}$$

We have

$$
(16)\qquad
\begin{aligned}
F^{++}\cdot F^{++} &= \tfrac{1}{4}(K_W + K_Q + K_P + K_E),\\
F^{+-}\cdot F^{+-} &= \tfrac{1}{4}(K_W + K_Q - K_P - K_E),\\
F^{-+}\cdot F^{-+} &= \tfrac{1}{4}(K_W - K_Q + K_P - K_E),\\
F^{--}\cdot F^{--} &= \tfrac{1}{4}(K_W - K_Q - K_P + K_E),\\
F^{++}\cdot F^{+-} = F^{++}\cdot F^{-+} &= F^{++}\cdot F^{--} = F^{+-}\cdot F^{-+}\\
&= F^{+-}\cdot F^{--} = FF^{-+}\cdot F^{--} = 0.
\end{aligned}
$$

Using these identities we can easily obtain

Theorem 2. *If A, B, C, D are arbitrary constants and not all zero, then*

$$
(17)\qquad
\begin{aligned}
I_W = \frac{1}{A^2+B^2+C^2+D^2}\Big\{ &\int_M (AF^{++} + BF^{+-} + CF^{-+} + DF^{--})^2 dV\\
&+ \big[(-I_P - I_Q - I_E)A^2 + (I_P + I_E - I_Q)B^2\\
&+ (I_E - I_P + I_Q)C^2 + (I_Q + I_P - I_E)D^2\big]\Big\}.
\end{aligned}
$$

Further, we have

Theorem 3.

$$
(18)\qquad
\begin{aligned}
\text{(a)}\quad I_W \geqslant \max\{&(-I_P - I_Q - I_E), (I_P + I_E - I_Q), (I_E - I_P + I_Q),\\
&(I_P + I_Q - I_E)\}.
\end{aligned}
$$

(b) *The equality sign of the above inequality holds if and only if at least one of F^{++}, F^{+-}, F^{-+}, F^{++} is zero.*

Proof. (a) follows directly from the identity (17).

(b) Suppose the equality sign holds, say $I_W = -I_P - I_Q - I_E$. Let $A = 1$. Then $B = C = D = 0$, and we have $F^{++} = 0$. Conversely, if $F^{++} = 0$, the same set of A, B, C, D gives $I_W = -I_P - I_Q - I_E$. Moreover, from (18) it follows that

$$
\begin{aligned}
-I_P - I_Q - I_E &\geqslant I_P + I_E - I_Q,\\
-I_P - I_Q - I_E &\geqslant I_E - I_P + I_Q,\\
-I_P - I_Q - I_E &\geqslant I_P + I_Q - I_E,
\end{aligned}
$$

and hence (b) is proved.

Now we can list several cases in which the Yang-Mills functional attains its possible minimal value for given values of I_P, I_E and I_Q:

No	F	I_P, I_Q, I_E	I_W
1	$F^{++}=0$	$-I_Q - I_E > 0, -I_P - I_E > 0, -I_P - I_Q > 0$	$-I_P - I_Q - I_E$
2	$F^{+-}=0$	$I_P + I_E > 0, -I_Q + I_P > 0, -I_Q + I_E > 0$	$I_P + I_E - I_Q$
3	$F^{-+}=0$	$I_E + I_Q > 0, I_Q - I_P > 0, I_E - I_P > 0$	$I_E + I_Q - I_P$
4	$F^{--}=0$	$I_P + I_Q > 0, I_P - I_E > 0, I_Q - I_E > 0$	$I_P + I_Q - I_E$
1,2	$F^{++}=F^{+-}=0$	$-I_Q > I_E = -I_P > I_Q$	$-I_Q$
1,3	$F^{++}=F^{-+}=0$	$-I_P > I_E = -I_Q > I_P$	$-I_P$
1,4	$F^{++}=F^{--}=0$	$-I_E > I_P = -I_Q > I_E$	$-I_E$
2,3	$F^{+-}=F^{-+}=0$	$I_E > I_P = I_Q > -I_E$	I_E
2,4	$F^{+-}=F^{--}=0$	$I_P > I_E = I_Q > -I_P$	I_P
3,4	$F^{-+}=F^{--}=0$	$I_Q > I_P = I_E > -I_Q$	I_Q
2,3,4	$F^{+-}=F^{-+}=F^{--}=0$	$I_E = I_P = I_Q > 0$	I_P
1,3,4	$F^{++}=F^{-+}=F^{--}=0$	$-I_P = I_Q = -I_E > 0$	$-I_P$
1,2,4	$F^{++}=F^{+-}=F^{--}=0$	$-I_Q = -I_E = I_P > 0$	I_P
1,2,3	$F^{++}=F^{+-}=F^{-+}=0$	$-I_Q = -I_P = I_E > 0$	$-I_P$

In a general manifold the values of I_E, I_P, I_Q place some constraints on both the Riemannian metric and the connection. For compact manifolds without boundary, I_E and I_P are topological invariants. So the cases (1,3), (1,4), (2,3) (2,4) (and hence (2,3,4) (1,3,4) (1,2,4), (1,2,3)) are more interesting than the other cases. In particular, consider the tangential bundle E over an oriented compact manifold M without boundary. If M admits an Einstein metric or a conformally flat metric with zero scalar curvature, then we have cases (2,3) or (1,4) respectively. Hence the Yang-Mills function attains absolute minimum in both cases.

Corollary. *The Einstein metric or the conformally flat metric with zero scalar curvature on the compact manifold without boundary minimizes the Yang-Mills functional on the tangential bundle, provided the connection is the Christoffel connection of the metric.*

Remark. If the Lie algebra g is decomposable to a direct sum of more than two components, then more detail results can be obtained in the same way.

References

[1] C. H. Gu, H. S. Hu, D. Q. Li, C. L. Shen, Y. L. Xin & C. N. Yang, *Riemannian spaces with local duality and gravitational instantons*, Scientia Sinica **21** (1978) 475.

[2] C. N. Yang, *Integral formalism for gauge fields*, Lett. Phys. Rev. **33** (1974) 445–447.

[3] S. S. Chern, *Geometry of characteristic classes*, Proc. 13th Biennial Sem. Canadian Math. Congress, Vol. 1, 1972, 1–40.

State University of New York, Stony Brook
Fudan University, Shanghai